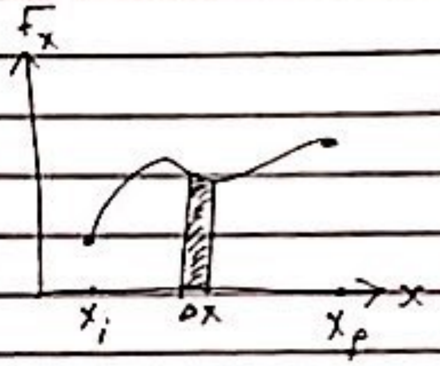
Work done by a varying force (1-D case)

The work done on an infinitesimal displacement Δx is



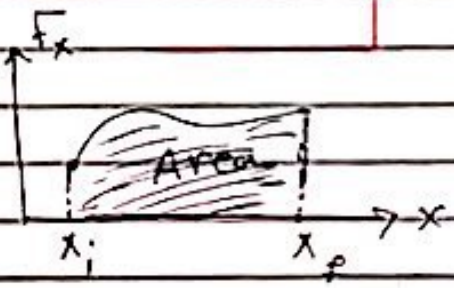
$$\Delta W = F_x \Delta x = \Delta A \quad \text{(area)}$$

The total work

$$W \sim \sum \Delta W = \sum_{x_i}^{x_f} F_x \Delta x$$

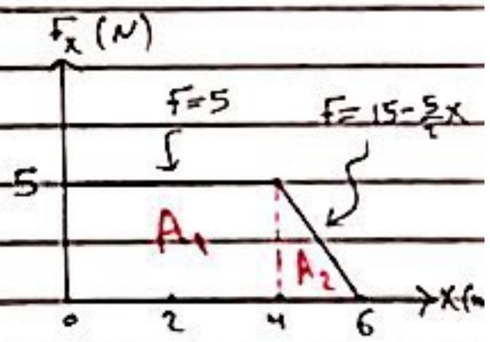
$$W = \lim_{\Delta x \to 0} \Delta W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx$$

$$\boxed{W = \int_{x_i}^{x_f} F_x \, dx = \text{Area under the curve}}$$



Example

Calculate the work done by the force F_x as the object moves from $x_i = 0$ to $x_f = 6$ m



solution

$$W = W_1 + W_2 = A_1 + A_2$$
$$= (5)(4) + \left(\tfrac{1}{2}\right)(2)(5) = 25 \text{ J}$$

or

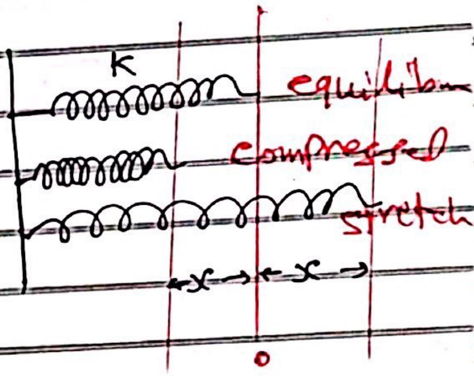
$$W = \int_0^6 F_x \, dx = \int_0^4 5 \, dx + \int_4^6 \left(15 - \tfrac{5}{2}x\right) dx = 20 + 5 = 25 \text{ J}$$

Work done by a spring

The force done by a spring stretched or compressed a small distance x from equilibrium is given by

$$F_s = -kx \quad \text{(Hooke's law)}$$

where k is the spring constant



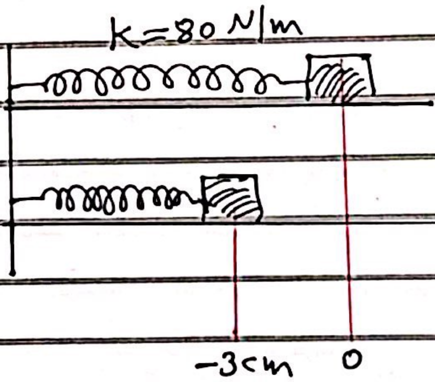
The work done by the spring force is given by

$$W_s = \int_{x_i}^{x_f} F_x \, dx = \int F_s \, dx = \int_{x_i}^{x_f} -kx \, dx$$

$$\boxed{W_s = \frac{1}{2} k \left(x_i^2 - x_f^2\right)}$$

Example

Calculate the work done by the spring force as the block moves from $x_i = -3$ cm to $x_f = 0$



$$W_s = \frac{1}{2} k \left(x_i^2 - x_f^2\right)$$

$$= \frac{1}{2}(80)\left[(-0.03)^2 - 0\right] = 3.6 \times 10^{-2} \text{ J}$$

Work and Kinetic Energy

□ 1-D motion

$$W = \int_{x_i}^{x_f} F_x \, dx$$

but $F_x = ma_x = m \dfrac{dv_x}{dt} = m \dfrac{dv_x}{dx} \dfrac{dx}{dt}$ (chain rule)

$$F_x = m \frac{dv_x}{dx} v_x$$

or $$F_x = m v \frac{dv}{dx}$$

$$W = \int \left(m v \frac{dv}{dx} \right) dx = \int_{v_i}^{v_f} m v \, dv$$

$$W = \frac{1}{2} m (v_f^2 - v_i^2)$$

We define the kinetic energy K as

$$K = \frac{1}{2} m v^2$$

$$\Rightarrow \boxed{W = K_f - K_i = \Delta K}$$ (Work-energy theorem)

□ 3-D motion

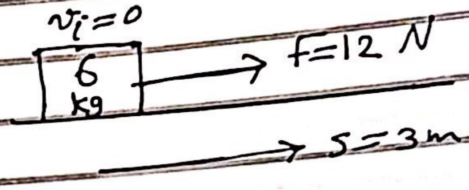
$$W = \int \vec{F} \cdot d\vec{s} = \int F_x \, dx + \int F_y \, dy + \int F_z \, dz$$

$$W = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

where
$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

Example

Find the speed of the block after it moves a distance 3 m on a smooth surface



$$W = FS = (12)(3) = 36\,J$$

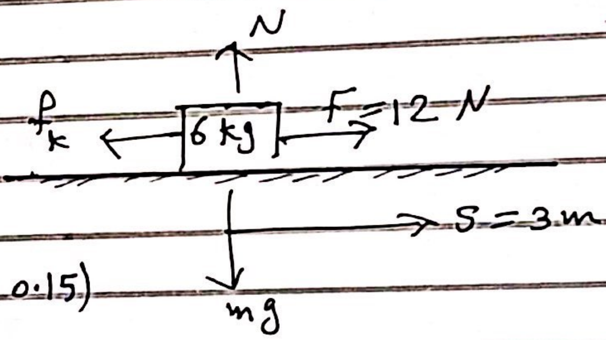
$$W = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

$$36 = \frac{1}{2}(6)\left[v_f^2 - 0\right]$$

$$v_f = 3.46 \text{ m/s}$$

Example

in the previous example, what is the speed of the block after it moves a distance 3 m on a rough surface ($\mu_k = 0.15$)



$$W_F = FS = (12)(3) = 36\,J$$

$$W_f = -f_k S = -\mu_k N S = -\mu_k m g S$$
$$= -(0.15)(6)(9.8)(3) = -26.5\,J$$

$$W_{net} = W_F + W_{f_k} = 36 - 26.5 = 9.5\,J$$

$$W_{net} = \Delta K = \frac{1}{2} m (v_f^2 - v_i^2)$$

$$9.5 = \frac{1}{2}(6)\left[v_f^2 - 0\right]$$

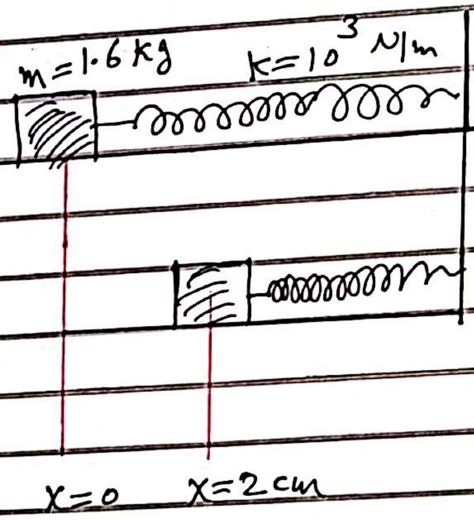
$$v_f = 1.78 \text{ m/s}$$

Example

After the spring is compressed a distance of 2 cm to the right, the block is released from rest. calculate the speed of the block as it passes through the equilibrium position $x=0$

a) if the surface is frictionless

b) if a constant frictional force of 4 N retards its motion



solution

a) $W_s = \frac{1}{2}k(x_i^2 - x_f^2) = \frac{1}{2}(10^3)[(0.02)^2 - 0] = 0.2\ J$

$W_s = \Delta K = \frac{1}{2}m(v_f^2 - v_i^2)$

$0.2 = \frac{1}{2}(1.6)[v_f^2 - 0]$

$v_f = 0.5\ m/s$

b) $W_s = 0.2\ J$

$W_f = -f_k S = -4(0.02) = -0.08\ J$

$W_{net} = W_s + W_f = 0.2 - 0.08 = 0.12\ J$

$W_{net} = \Delta K = \frac{1}{2}m(v_f^2 - v_i^3)$

$0.12 = \frac{1}{2}(1.6)[v_f^2 - 0]$

$v_f = 0.39\ m/s$

Example

As an object moves a distance d upward an inclined rough plane,

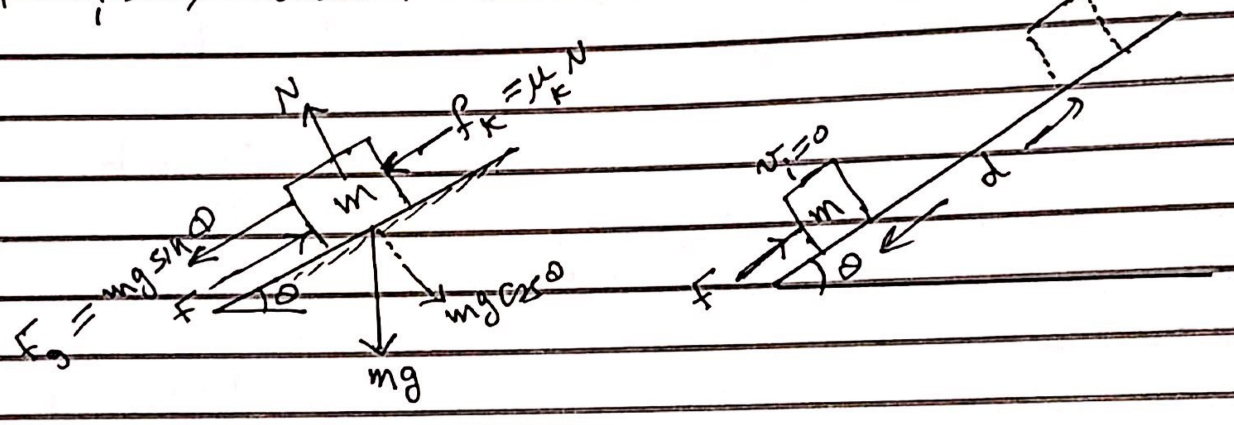
a) calculate the work done by the applied force F

b) " " " " " " force of gravity F_g

c) " " " " " " frictional force f_k

d) find the net work

e) if $v_i = 0$, what is the final speed of the object



a) $W_F = Fd\cos(0) = Fd$

b) $W_g = (mg\sin\theta)(d)\cos 180 = -mgd\sin\theta$

c) $W_f = f_k d\cos 180 = -f_k d = -\mu_k mg\cos\theta\, d$

d) $W_{net} = Fd - mgd\sin\theta - \mu_k mgd\cos\theta$

e) $W_{net} = \Delta K = \frac{1}{2}m(v_f^2 - v_i^2) = \frac{1}{2}mv_f^2$

$$v_f = \sqrt{\frac{2}{m}W_{net}}$$

$$= \sqrt{\frac{2d}{m}\left[F - mg\sin\theta - \mu_k mg\cos\theta\right]}$$

* if $F = 15\,N$, $d = 1\,m$, $\theta = 25°$, $m = 1.5\,kg$, $\mu_k = 0.3$

then $W_F = 15J$, $W_g = -6.2J$, $W_f = -4J$, $W_{net} = 4.8J$

$v_f = 2.5\,m/s$